USING VISUAL REPRESENTATIONS IN TEACHING TO DEVELOP MATHEMATICAL COMMUNICATION COMPETENCE FOR STUDENTS IN VIETNAM

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Abstract

The role of representations in the teaching and learning mathematics has evolved considerably in recent years. Mathematical communication competence is one of standards for school mathematics. In this paper: we deals with a lot of visual representations, which were performed in an actual mathematics classroom and could help students to think and illustrate pure symbolic mathematical solutions. These representations were used by students and were an effective tool in communicating mathematics to their teacher and classmates. We also interested which visual representations are really useful in mathematics education and promote students to study actively. Through experimental lesson, the paper shows that visual representations provide opportunities for students to visualize the problem and assist in problem solving.

Keywords: visual representations, mathematical communication.

1. Introduction

In recent years, views on the role of representation in teaching and learning mathematics are being studied; some researchers have emphasized representation in mathematical fields by researches, such as Interger (Carraher, 1990), Functions (Even, 1998), Geometry (Mesquita, 1998). Some Other people have studied process, in which, students work in groups in order to build representations (Davis and Maher, 1997). Vui (2009) focused on analyzing the remarkable contributions of visual representation into supporting students learning mathematics at high schools, in order to help teachers making good decisions and deciding to choose which kind of representation is effective in Maths lessons by themselves.

Phuong (2015) focused on research that develops inductive and extrapolation ability in order for students to find the mathematical rules by the visual representations. This research focus on Mathematical communication ability of students by using visual representations of students to illustrate pure mathematical phenomena, so students could connect Maths with the real life and they contribute the mathematical knowledge.

2. Literature review

In this part, we present the results of the research on mathematical representations that assists students in expressing mathematical communication which is suitable for the statement "students in classroom created and used representations to organize, record, and communicate mathematical ideas; select, apply, and translate between mathematical representations to solve problems and use representa-
tions to model and explain the physical, social, and mathematical phenomena" (NCTM, 2000). Then, we illustrate examples of teaching situations with using mathematical representations.

2.1. Forms of mathematical representations
Bruner focused on the study of children's mathematical awareness as well as on representation thinking, he pointed out that it is possible to divide the representation into three categories from low to high as following (Vui, 2009):
- Reality: the actual representations of the lowest level, and by hand;
- Imagery: visual representations using images, graphs, charts, tables ...;
- Symbols: include language and symbol representations.

Tadao (2007) classifies representations in math education into five more specific forms as following:
- Realistic representation: Representations based on the actual state of the object. This type of representation can be directly, specific and natural effects.
- Manipurative representation: they are teaching aids tools, replacement or imitation of objects that students can affect directly. This type of representation can be very specific and artificial.
- Visual representation: Representation using illustrations, diagrams, graphs, charts. This is a kind of visual and lively representation.
- Language representation: These representations use pure language to express (say or write). This type of representation is governed by conventions, but lacking in succinctness; On the other hand, this representation is descriptive and can create a sense of familiarity.
- Represented by algebraic symbols: Representations using mathematical symbols such as numbers, letters, symbols.

2.2. Mathematical Communicating Ability
In this section, we explain some of the terms mathematical communication and the ability to communicate mathematics. Mathematical communication is a form of communication in which one tries to convince others of his or her ideas, thoughts, questions or mathematical hypotheses to share ideas and clarify understanding, know about these mathematical problems. Through discussion and questioning, mathematical ideas are: reflections, discussions and corrections. The process of student reasoning and analysis systematically helps children to strengthen their knowledge and understanding of math. Through communication, students solve problems more effectively, can explain mathematical concepts and have problem solving skills (Lim, 2008).

The ability to communicate in mathematics: including expressing one's own view of mathematical problems, understanding the ideas of others when presenting the problem, expressing their ideas clearly, using mathematical language, conventions and mathematical symbols (Pham Gia Duc and Pham Duc Quang, 2002; Mónica Miyagui, 2007).

3. Research Methodology
We applied the qualitative research that consists of the designed experiment and the participant's observation method. Experimental teaching was conducted in the year 2013–2014 and retaught in the year 2016–2017 at Saigon High School, District 5, Ho Chi Minh city. There are 230 students including two classes for the grade 8 (80 students), two classes for the grade 6 (80 students) and two classes for the grade 11 (70 students). The data are presented here, as evidence of students' arguments, students' mathematical reasoning. Data analysis is qualitative.

4. Finding
4.1. Example for teaching situation that use mathematical representations
Over the past decade, math educators have found mathematical representations as "the useful tools for communicating in information and understanding of students" (NCTM, 2000). Mathematical representation is at the heart of understanding about a mathematical concept and problem-solving activity of a person; the flexible using of could make more meaningful and more effective in studying Maths. Representation perception is the core of studying mathematics that is reflected clearly in recent reform
efforts, especially in the United States. The flexible using of mathematical representations help students have the knowledge exactly, thereby contributing to support students express their mathematical ideas.

**The first experiment**: The problem in the Vietnamese eighth grade mathematics textbook is to compare the area of a square and a rhombus of the same circumference. We illustrate this problem by mathematical representations.

- **Language representation**: Give a rhombus and a square of the same circumference. Which shape has larger area? Why?
- **Visual representation**: On grid paper (each square has an area of 1).

**Teacher**: Chose square and rhombus of sides 5 as shown in Figure 1 for the purpose of computing the height of rhombus being an integer.

**Students**: By observing Figure 1, the students explain the reasons why the quadrilaterals are the rhombus and the square of the same circumference and the area of rhombus is smaller than the area of square, as illustrated in Table 1.

![Figure](https://example.com/figure.png)

Figure **Error! No text of specified style in document.**. The square and the rhombus have sides with a length of 5.

<table>
<thead>
<tr>
<th>Draw DJ which is perpendicular to DJ = 3; AJ = 4. Apply Pythagorean theorem in the right-angled triangle ADJ, calculate AD = 5. From this, the ABCD quadrilateral is a rhombus of sides 5, the circumference is 20 and the area of the ABCD is 15.</th>
<th>Draw HK which is perpendicular to EF, EK = 4; HK = 3. Apply Pythagorean theorem in the right-angled triangle EHK, calculate EH = 5. From this inference, the EFGH quadrilateral is a rhombus of sides 5, the circumference is 20 and the area of the EFGH rhombus is 20.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The MNPQ square has a circumference of 20 and an area of 25.</td>
<td>100% of the students applied the visual representations on the paper, so from the observation, students find the result of the problem fastly by arguments, informal proofs.</td>
</tr>
</tbody>
</table>

- **Manipulative representation**:
  - **Teacher**: Using the Geometer's sketchpad (GSP) software, draw squares and rhombuses of the same circumference.
  - **Students**: Using the functionality of the GSP software, measure the perimeter and area of the images.
Students: Through observation, 100% of the students found out the solution. The next, they prove that comment “If a rhombus and a square of the same circumference then square has larger area” is true. 80% of the students reused the result in the Figure 1, they found out the length of the height of the rhombus is $DJ$, $DJ$ is always less than the side of rhombus so $S_{ABCD} = DJ \cdot AB$ and $S_{ABCD} < AB \cdot AB$ then a rhombus and a square of the same circumference then square has larger area.

4.2. Example of using the visual representations for communicating mathematics

We designed the experiments according to students’ ability, students use visual representations to communicate mathematics with peers and teachers as well as demonstrate their ability to think and re-solve problem.

The second experiment: The problem in the Vietnamese eighth grade mathematics textbook is to factorise the following polynomial $2x^2 + 7x + 6$. In Vietnam, factorising a polynomial is a mathematical form for grade 8. Factorising a polynomial is not easy for student to do this theme. So, we design Experiment 1th as following: Father of Vi has some rectangular and square cardboard as in Figure 3a. He asks Vi to fold all the cardboard into a rectangular board. According to you, How did Vi arrange? Write a formula that shows the area of the rectangle. From there, factorise the following polynomial $2x^2 + 7x + 6$.

Analyze the second experiment:

Through the experiment, the arrangement of squares, rectangles (in Figure 3a) into a rectangle, 100% of the students can do (Figure 3b). Through realistic situations, students have the opportunity to attach an abstract algebra concept to the visual imagery of the area, thereby helping students solve math problems easily. Specifically: Observe the rectangles in Figure 3b, each of which is different from Figure 3a, but each rectangle has a breadth, length, area, respectively là $x+2$, $2x+3$, $2x^2 + 7x + 6$. Combined using the rectangular area formula we have $2x^2 + 7x + 6 = (2x+3)(x+2)$.

From there, student factorise the following polynomial $2x^2 + 7x + 6$ easily.
Then the teacher asks to factorise the following polynomial $2x^2 - 7x + 6$.

Through experiments, we have obtained the following results:

- Students tried to write the following polynomial $2x^2 - 7x + 6$ in the form of a product of polynomials by different ways, for example:

  $2x^2 - 7x + 6 = 2x^2 - 4x - 3x + 6 = 2x(x - 2) - 3(x - 2) = (2x - 3)(x - 2)$; (40% of the students)

  $2x^2 - 7x + 6 = 2x^2 - 8 - 7x + 14 = 2(x^2 - 4) - 7(x - 2) = 2(x - 2)(x + 2) - 7(x - 2) = (x - 2)(2x - 3)$

  (40% of the students).

Students use pocket calculators to find solutions of polynomial $2x^2 - 7x + 6$ then factorise the polynomial $2x^2 - 7x + 6 = (2x - 3)(x - 2)$ (40% of the students).

- Teacher suggested: Can we use the same visual representation as figure 3a to factorise the polynomial? 40% of the students answered yes and they explained that change $x$ in Figure 3a to $(-x)$ in figure 3c, then, $2x^2 - 7x + 6 = (3 - 2x)(2 - x)$.

**The third experiment:**

a) Observe figure 4a, showing how to find $S_n; S_n$ with $n \in N$.

b) Calculate the sum $S = 1 + 2 + 3 + ... + n$ with $n \in N$.

**Figure 4a**

**Analyze the third experiment:**
a) We have obtained some results as follows:

The first solution: Observe the number of dots in Figure 4a, 100% of the students comment:

\[ S_1 = 1; S_2 = 1 + 2; S_3 = 1 + 2 + 3; S_4 = 1 + 2 + 3 + 4; S_5 = 1 + 2 + 3 + 4 + 5; \]

so \[ S_5 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45. \] Similarly, we have \[ S_n = 1 + 2 + 3 + \ldots + n \]

Another solution: Observe the dots in Figure 4a, 50% students comment:

\[ S_1 = 1; S_2 = 1 + 2; S_3 = 1 + 2 + 3; S_4 = 1 + 2 + 3 + 4; S_5 = 1 + 2 + 3 + 4 + 5; \]

so \[ S_5 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45; \] Similarly, \[ S_n = 1 + 2 + 3 + \ldots + n \]

b) 100% of the students found that \[ S = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \] by using the formula learned in elementary school. But when we ask the students to explain, 40% of the students explained as follows:

\[ S = 1 + 2 + 3 + \ldots + (n-1) + n \text{ or } S = n + (n-1) + \ldots + 3 + 2 + 1 \]

then \[ S + S = (1 + 2 + 3 + \ldots + n) + \left[ n + (n-1) + \ldots + 3 + 2 + 1 \right] \]

so \[ 2S = \left[ (1 + n) + (2 + n-1) + \ldots + (n + 1) \right] \text{ so } 2S = n(n+1) \text{ so } S = \frac{n(n+1)}{2} \]

To help students to simplify the sum \[ S = 1 + 2 + 3 + \ldots + n \] with \[ n \in \mathbb{N} \] by using the visual representation, the teacher showed the students how to observe figure 4b and then calculated the sum S.

![Figure 4b](image)

The fourth experiment: Give the following number sequence: 5, 9, 13, 17, 21, 25, ...

a) Look for the 8th and nth terms.

b) Look for visual representation illustrations for the above sequence.

Analyze the fourth experiment:

a) We collected the results: 100% of the students found the 8th term is 73; 60% of the good students found the nth term is \[ 4n + 1 \] with \[ n \in \mathbb{N}. \]

b) In general, students are not used to the use of visual representation. Interestingly, through visual representation students can detect the form of the nth term.
Visual representation 1st (see Figure 5a) is proposed by 60% of the students and 100% of the students sets the general number of dots.

\[
\begin{align*}
  a_1 &= 5 \\
  a_2 &= 9 \\
  a_3 &= 13 \\
  a_4 &= 17 \\
\end{align*}
\]

Figure 5a

\[
a_1 = 4.1 + 1; a_2 = 4.2 + 1; a_3 = 4.3 + 1; a_4 = 4.4 + 1; \ldots; a_n = 4.n + 1.
\]

Visual representation 2nd, 3rd, 4th (see Fig. 5b, 5c, 6a, 6b) proposed by the teacher, 100% of the student argues for the \( n \)th term.

Visual representation 2nd of the number of segments (see Figure 5b):

\[
\begin{align*}
  b_1 &= 5 \\
  b_2 &= 9 \\
  b_3 &= 13 \\
  b_4 &= 17 \\
\end{align*}
\]

Figure 5b

Students argue as follows:

\[
b_1 = 5 \quad \text{(1 picture with 5 straight lines)};
\]

\[
b_2 = 5.2 - 1 \quad \text{(There are 2 pictures where each picture has 5 straight lines but one line is repeated)};
\]

\[
b_3 = 5.3 - 2 \quad \text{(There are 3 pictures with 5 straight lines but 2 straight lines are repeated)};
\]

\[
b_4 = 5.4 - 3 \quad \text{(there are 4 pictures with 5 straight lines but 3 straight lines are repeated)};
\]

So \( b_n = 5n - (n - 1) = 4n + 1 \)

Visual representation 3 of the number of vertices (see Figure 5c):

\[
\begin{align*}
  c_1 &= 5 \\
  c_2 &= 9 \\
  c_3 &= 13 \\
  c_4 &= 17 \\
\end{align*}
\]

Figure 5c

Students argue as follows:

\[
c_1 = 5 \quad \text{(1 star has 5 vertices)};
\]

\[
c_2 = 5.2 - 1 \quad \text{(two stars but one vertice repeated)};
\]

\[
c_3 = 5.3 - 2 \quad \text{(with 3 stars but repeating 2 vertices)};
\]

\[
c_4 = 5.4 - 3 \quad \text{(4 stars but repeating 3 vertices)};
\]

So \( c_n = 5n - (n - 1) = 4n + 1 \)

Visual representation of triangular numbers (see Figure 6a):


Add the missing triangle in Figure 6a to Figure 6b

Students argue as follows:

$e_1 = 8 = 2.4$ (there are 2 rectangles that each contain 4 triangles)

Because $d_1 = e_1 - 3$ (additional 3 triangles) so $d_1 = 2.4 - 3$

$e_2 = 12 = 3.4$ (there are 3 rectangles that each contains 4 triangles)

Because $d_2 = e_2 - 3$ (additional 3 triangles) so $d_2 = 3.4 - 3$

$e_3 = 16 = 4.4$ (there are 4 rectangles that each contains 4 triangles)

Because $d_3 = e_3 - 3$ (additional 3 triangles) so $d_3 = 4.4 - 3$

$e_4 = 20 = 5.4$ (there are 5 rectangles that each contains 4 triangles)

Because $d_4 = e_4 - 3$ (additional 3 triangles) so $d_4 = 5.4 - 3$

Hence, $d_n = (n + 1)4 - 3 = 4n + 1$

5. Discussion & Conclusion

5.1. Discussion

Visual representation is a tool that helps students capture abstract concepts based on specific representation in mathematics instruction. This is in line with the "From visual representation to abstract thinking" (please to see the first experiment).

Students: On grid paper (each square has an area of 1). We can draw a square of sides 5 but we can not draw a zonbus of sides 5. We calculate the area of square and zonbus easily; especially, area of square and zonbus is interger number. We can reused the result in the Figure 1 to prove the statement “If the rhombus and the square of the same circumference then the area of rhombus is smaller than the area of square”.

When teacher encourages students to express their mathematical ideas through mathematical communication, teachers understand what students are learning (please to see the second experiment).
In the second experiment, 60% of the students answered no when the teacher asked “Can we use the same visual representation as figure 3a to factorise the polynomial $2x^2 - 7x + 6$?”. Because they explained that $x$ in figure 3a is positive, so it is not possible to do the same thing. Through the students’ point of view, we find that $x$, $-x$ is abstract for students, most of them have a mistake that $x$ is positive and $-x$ is negative, because they have inferred similar such as $3 > 0$ and $-3 < 0$. The visual representation assists the student in finding the way to factorise the following polynomial $2x^2 + 7x + 6$, $2x^2 - 7x + 6$.

Using visual representations to communicate students’ ideas and mathematical results supports students be interesting in learning (please to see the third experiment).

According to figure 4b, 80% of the students commented the sum $S=1+2+3+...+n$ with $n \in \mathbb{N}$ corresponding as a half of the dots sum within the rectangle having the width and length respectively $n$ and $(n+1)$. In addition, we found one thing: the same number is illustrated by different symbols such as the number of dots or the number of segments or the number of vertices or triangles that lead students to argue how to set the total number differently.

5.2. Conclusion

Through the above mentioned experiments, the success of the experiments is:

- Students work directly on mathematical representations and observe visual representations so they predict mathematical results.
- During the process of observing and predicting the results of inductive, students have boldly talk with their friends about their thinking, have the opportunity to communicate and debate with teachers and friends.
- Students have the opportunity to express creative thinking and critical thinking in the process of solving problems with visual representation.
- Students recognize that there is a deep connection between represented by algebraic symbols and visual representations. That connection helps student to think and help students understand situations as well as capture abstract concepts.

Through the above mentioned experiments, the difficulty of the experiments is:

- The Vietnamese lesson’s content links to visual representations is very fewer and teachers should choose lesson’s content carefully to develop mathematical communication competence.
- An important point, the Vietnamese students are not used to create visual representations.

5.3. Suggestion

In my opinion, using visual representations should be considered and implemented in the classroom to improve the student's mathematical communication competence.

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